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The Conservation of Source Strength in Linear Field Theories and the Superpotentials

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Abstract

In linear field theories for vector potentials A_i and tensor potentials $g_{ik} = g_{ki}$, the Maxwell and the linearized Einstein equations are the only field equations from which true conservation laws result for each gauge of the field equations.

In linear field theories one can derive pseudoconservation laws by means of a superpotential, just as in Einstein's theory of gravitation. We shall see that these superpotentials of linear field theories give some deeper insight about the meaning of the several superpotentials of gravitational theory.

Let us start with pseudoconservation laws for a "current vector" S_i in linear field theory. We denote by $\Box$ the linear d'Alembert operator

$$\Box = \eta^{ik} \partial_i \partial_k \quad (i, k = 0, 1, 2, 3) \tag{1}$$

The linear field equations may be given by this wave operator $\Box$, operating on the vector field A_i, and the source strength J_i, generating the field:

$$\eta^{lm} \partial_l \partial_m A_i \equiv A_{i,ll} = \mathcal{J}_i \tag{2}$$

Here the left-hand side of Eq. (2) cannot be written as a divergence of an antisymmetric superpotential. Using the field equations, we can now define a "current vector" S_i in the following way:

$$S_i = A_{l,li} - \mathcal{J}_i = A_{l,li} - A_{i,ll} \tag{3}$$

One sees that the divergence of S_i vanishes identically:

$$S_{i,i} = 0 \tag{4}$$

and S_i can be derived from the antisymmetric superpotential

$$F_{il} = A_{l,i} - A_{i,l} = -F_{li} \tag{5}$$

according to

$$S_i = F_{il,l} \tag{6}$$

It is well known that the superpotential F_{il} is gauge invariant under gauge transformations

$$A_l' = A_l + \phi_{,l} \tag{7}$$

In a similar way we can obtain pseudoconservation laws for a symmetric tensor field g_{ik}, fulfilling the linear wave equation:

$$\Box g_{ik} \equiv g_{ik,ll} = T^*_{ik} \tag{8}$$

where the symmetric tensor T^*_{ik} is the source strength, generating the field, and again $\Box g_{ik}$ is not the divergence of an antisymmetric superpotential.

Just as in the case of the vector field A_i we can define the following "tensors of matter," which are conserved:

$$S^{\mathrm{I}}_{ik} = g_{il,lk} - T^*_{ik} \tag{9}$$

$$S^{\mathrm{II}}_{ik} = g_{kl,li} - T^*_{ik} \tag{10}$$

Using the field Eqs. (8), one verifies quickly that Eqs. (9) and (10) indeed yield

$$S^{\mathrm{I}}{}_{ik,k} = 0; \; S^{\mathrm{II}}{}_{ik,i} = 0 \tag{11}$$

Further, one gets the more explicit expressions for $S^{\mathrm{I}}{}_{ik}$ and $S^{\mathrm{II}}{}_{ik}$:

$$S^{\mathrm{I}}_{ik} = g_{il,lk} - g_{ik,ll} \tag{12}$$

$$S^{\mathrm{II}}_{ik} = g_{kl,il} - g_{ik,ll} \tag{13}$$

The expressions (12) and (13) show that the corresponding superpotentials are

$$\mathfrak{U}^{\mathrm{I}}{}_{ikl} = g_{il,k} - g_{ik,l} = -\mathfrak{U}^{\mathrm{I}}{}_{ilk} \tag{14}$$

and

$$\mathfrak{U}^{\mathrm{II}}{}_{kil} = g_{kl,i} - g_{ik,l} = -\mathfrak{U}^{\mathrm{II}}{}_{kli} \tag{15}$$

so that

$$S^{\mathrm{I}}{}_{ik} = \mathfrak{U}^{\mathrm{I}}{}_{ikl,l}, \; S^{\mathrm{II}}{}_{ik} = \mathfrak{U}^{\mathrm{II}}{}_{kil,l} \tag{16}$$

The most general antisymmetrical superpotential for our linear field theory we can easily obtain if we take the most general superpotential of the nonlinear theory, given by Møller (1958), $\mathfrak{M}_i{}^{kl}$. We take its linearized form and add it to one of our linear superpotentials $\mathfrak{U}_i{}^{\mathrm{I}\,kl}$ or $\mathfrak{U}_i{}^{\mathrm{II}\,kl}$. (The indices k, l are raised with the Minkowski metric η^{kl}, of course.) If we take $\mathfrak{U}_i{}^{\mathrm{I}\,kl}$, we get the following expression for the general superpotential in linear tensor field theory:

$$\mathfrak{D}_i{}^{kl} = - \mathfrak{D}_i{}^{lk} = \mathfrak{U}_i{}^{kl} + \mathfrak{M}_i{}^{kl} = (1 + a_1) g_{in,m}(\eta^{km}\eta^{ln} - \eta^{lm}\eta^{kn})$$
$$+ a_2(\delta_i{}^k g^{lm}{}_{,m} - \delta_i{}^l g^{kn}{}_{,n}) + a_3(\delta_i{}^k \eta^{lm} - \delta_i{}^l \eta^{km}) g_{rr,m} \tag{17}$$

Now true conservation laws mean (in linear field theory), that the sources of the field J_i and T_{ik} respectively, are divergence free. But in this case the wave operators WA_i and Wg_{ik}, respectively, must already have vanishing divergence, because of the field Eqs. (2) and (8). That means that these wave operators can be represented as the divergence of the superpotential, and so the sources J_i and T_{ik} themselves are the divergence of the superpotential.

In the case of the vector field A_i we get in this manner Maxwell's equations:

$$WA_i \equiv F_i{}^k{}_{,k} = \mathcal{J}_i \tag{18}$$

In the case of the tensor-field g_{ik} we have additionally to postulate the symmetry of T_{ik}. This gives us the linearized Einstein equations:

$$Wg_{ik} \equiv g_{ik,mm} + g_{mm,ik} - g_{im,km} - g_{km,im} - \eta_{ik} g_{rr,mm}$$
$$+ \eta_{ik} g_{rm,rm} = - 2T_{ik} \tag{19}$$

With Einstein's (1916a) gauge

$$\eta^{ik} g_{ik} = g_{rr} = \text{const} \tag{20}$$

Eq. (19) takes the form

$$g_{ik,mm} - g_{im,km} - g_{km,im} + \eta_{ik} g_{rm,rm} = - 2T_{ik} \tag{21}$$

The superpotential, which gives the left-hand side of Eq. (19), is just the (linearized) superpotential of Einstein (1916a) and v. Freud (1939):

$$\mathfrak{F}_{ikl} = - \tfrac{1}{2}(g_{ik,l} - g_{il,k}) - \tfrac{1}{2}(\eta_{ik} g_{lm,m} - \eta_{il} g_{km,m}) - \tfrac{1}{2}(\eta_{il} g_{rr,k}$$
$$- \eta_{ik} g_{rr,l}) \tag{22}$$

That means that the left-hand side of Eq. (19) results as the divergence of (17) with the constants a_1, a_2, a_3 chosen in the following manner:

$$1 + a_1 = a_2 = - a_3 = - \tfrac{1}{2} \tag{23}$$

We can write now (19) in the form

$$\mathfrak{F}_{ik}{}^l{}_{,l} = T_{ik} = T_{kl} \tag{24}$$

The second term of $\mathfrak{F}_{ikl}$ in (22) serves to symmetrize T_{ik}, defined by (24). The third term of (22) secures the gauge invariance of the field equation, which means the invariance of $\mathfrak{F}_{ik}{}^l{}_{,l}$ under gauge transformations of the form

$$g'_{ik} = g_{ik} + \xi_{i,k} + \xi_{k,i} \tag{25}$$

with $\xi_{r,r} \neq 0$.

It is just the (linearized) ansatz of Einstein (1916a) and v. Freud (1939) that produces the gauge invariance of the divergence of the general superpotential $\mathfrak{D}_i{}^{kl}{}_{,l}$ and the resulting field equations. If we take the general superpotential (17), and perform a gauge transformation (25), we get

$$\mathfrak{D}'_{ikl} = \mathfrak{D}'_{ikl}[g'_{ts}] = \mathfrak{D}_{ikl}[g_{ts}] + (1 + a_1)(\xi_{k,l} - \xi_{l,k})_{,i} + a_2[\eta_{ik}(\xi_{l,m} + \xi_{m,l})_{,m} - \eta_{il}(\xi_{k,m} + \xi_{m,k})_{,m}] - 2a_3[\eta_{il}\eta_{km} - \eta_{ik}\eta_{lm}]\xi_{r,rm} \tag{26}$$

The postulate of gauge invariance (which is equivalent to covariance of the field equations against infinitesimal transformations) of the divergence of the superpotential

$$\mathfrak{D}'_i{}^{kl}{}_{,l} = \mathfrak{D}_i{}^{kl}{}_{,l} \tag{27}$$

gives the following conditions for the constants a_1, a_2, a_3:

$$\begin{aligned} \xi_{k,lli}(1 + a_1 - a_2) &= 0 \\ \xi_{l,lik}(-1 - a_1 - a_2 - 2a_3) &= 0 \\ \eta_{ik}\xi_{l,lmm}(2a_2 + 2a_3) &= 0 \end{aligned} \tag{28}$$

From (28) it follows that

$$1 + a_1 = a_2 = -a_3 \tag{29}$$

which just defines the superpotential of Einstein and v. Freud (up to a common constant factor). Therefore, the linearized superpotential $\mathfrak{F}_i{}^{kl}$ of Einstein and v. Freud's superpotential $\chi_i{}^{kl}$ gives the only gauge invariant $\mathfrak{D}_i{}^{kl}{}_{,l}$, That means we could take the field equation

$$g_{ik,ll} - g_{il,kl} - g_{kl,il} + \eta_{ik}g_{rm,rm} + \beta(g_{rr,ik} - \eta_{ik}g_{rr,mm}) = T^*_{ik} \tag{30}$$

in linear tensor field theory; this equation would allow true conservation laws, but the postulate of gauge invariance tells us that only the field equations with $\beta = 1$ are possible.

The transition from linear tensor field theory to the theory of general relativity now follows from the remark that locally at a world point P_0 (with coordinates $X_0^i = 0$) in a system of geodesic coordinates yields for the metric g_{ik} and the Christoffel symbols

$$g_{ik}(0) = \eta_{ik}, \qquad \Gamma^i_{kl}(0) = 0 \tag{31}$$

In these coordinates Einstein's equations of the gravitational field (Einstein 1916a)

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -T_{ik} \tag{32}$$

go over in their linearized form, and simultaneously the dynamical equation

$$T_i{}^k{}_{,k} + \Gamma_r{}^k{}_k T_i{}^r - \Gamma_i{}^r{}_k T_r{}^k = 0 \tag{33}$$

takes the form of a conservation law for energy and impulse. Now we can see immediatley that the superpotential χ_i^{kl} of Einstein and v. Freud in general relatively theory is also distinguished as that superpotential that allows a true conservation law for energy and impulse. With χ_i^{kl} the dynamical equation takes the form

$$\chi_i^{kl}{}_{,lk} = (\sqrt{-g}\, T_i^k + \mathfrak{A}_i^k)_{,k} = 0 \tag{34}$$

where $\mathfrak{A}_i^k$ is Einstein's affine tensor of gravitational energy (Einstein, 1916a, 1916b), which is bilinear in the first derivatives of g_{ik}, and therefore vanishing in P_0. In P_0 the divergence of the superpotential of Einstein and v. Freud then equals

$$\chi_i^{kl}{}_{,l}(0) = \mathfrak{F}_i^{kl}{}_{,l} = T_i^k \tag{35}$$

that means that it is just the source density of the matter as in the case of linear field theory. The tensor of energy and impulse of the gravitational field $\mathfrak{A}_i^k$ does not occur as a source locally (cf. Treder, 1974).

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